Define $D_i^h u = \frac{u(x+he_i)-u(x)}{h}$, and let $u \in L^\infty(\Omega, \mathbb{R}^n) \cap W^{1,2}(\Omega, \mathbb{R}^n)$ be continuous, with $\Omega \subseteq \mathbb{R}^m$ solve

$$\int_\Omega B^k(u, \nabla u)v^k = \int_\Omega \sum_{i=1}^m u^k_{,i} v^k_{,i} \quad \forall v \in L^\infty(\Omega, \mathbb{R}^n) \cap W_0^{1,2}(\Omega, \mathbb{R}^n), \forall i = 1, ..., n \tag{1}$$

and $B^k = B^k(z,p)$ a quadratic form in p_k and $C^{p,\gamma}(\mathbb{R}^n)$ and bounded in z(i.e. $B^k = b^k_{ij}(z)p^i_k p^j_k$ so $B^k(u, \nabla u) = b^k_{ij}(u)u^i_{,k}u^j_{,k}$). In particular $B \in C^1(\mathbb{R}^n \times \mathbb{R}^{m\times n})$.
We will show the following:

Theorem 1. *Let $u \in L^\infty(\Omega, \mathbb{R}^n) \cap W^{1,2}(\Omega, \mathbb{R}^n)$ with $\Omega \subseteq \mathbb{R}^m$ a bounded domain solve (1). Then for each $U \subseteq \bar{U} \subseteq \Omega$ it holds that $u \in W^{2,2}(U)$.*

We will proceed in steps.
Define $\eta = \zeta^2$ as a cutoff function for U. Then for small enough $|h| > 0$ it holds that $w = D_i^{-h}(\eta^2 D_i^h u) \in L^\infty(\Omega, \mathbb{R}^n) \cap W_0^{1,2}(\Omega, \mathbb{R}^n)$.
Now, we shall investigate certain related linear equations.
Let $x \in \Omega$ with $d(x, \partial\Omega) > 2h$ and define

$$(1-t)(x_1, P_1) + t(x_2, P_2) = (c_1(t), c_2(t)) \in \mathbb{R}^n \times \mathbb{R}^{m\times n}$$

where $x_1 = u(x), x_2 = u(x + he_i)$ $P_1 = \nabla u(x), P_2 = \nabla u(x + he_i)$.

Lemma 2. *It holds that*

$$D_i^h B^k(u, \nabla u)(x) = \sum_{l=1}^n (D_i^h u^l) \int_0^1 b^k_{ij,l}(c_1(t))c_2^{ki}(t)c_2^{kj}(t)dt + \sum_{\alpha=1}^n (D_i^h u^\alpha_{,k}) \int_0^1 (b^k_{\alpha j} + b^k_{j\alpha})c_2^{kj}(t)dt$$

under the above hypotheses

Proof. By the fundamental theorem of calculus

$$B^k(x_2, P_2) - B^k(x_1, P_1) = \int_0^1 \frac{d}{dt} B^k((1-t)(x_1, P_1) + t(x_2, P_2))dt$$

Now, it holds that as $B \in C^1(\mathbb{R}^n \times \mathbb{R}^{m\times n})$ by the chain rule

$$\frac{d}{dt} B^k((1-t)(x_1, P_1) + t(x_2, P_2)) = \sum_{l=1}^n B^k_{x_l}(c(t))c^{kl}_{1,t}(t) + \sum_\alpha B^k_{p^\alpha_k}(c(t))c_2^{kj}(t)$$

On the other hand

$$B^k_{x_l}(c(t)) = b^k_{ij,l}(c_1(t))c_2^{ki}(t)c_2^{kj}(t)$$

$$B^k_{p^\alpha_k}(c(t)) = (b^k_{\alpha j} + b^k_{j\alpha})c_2^{kj}(t)$$

Substitute this and we are done. □

Now, we define $a^k_l = \eta^2 \int_0^1 b^k_{ij,l}(c_1(t))c_2^{ki}(t)c_2^{kj}(t)dt$ and $c^k_\alpha = \eta^2 \int_0^1 (b^k_{\alpha j} + b^k_{j\alpha})c_2^{kj}(t)dt$

Lemma 3. *With everything as above $D_i^h u$ solves the linear system*

$$\int_\Omega a_l^k u^l u^k + c_\alpha^k u_{,k}^\alpha u^k = \int_\Omega \sum_{j=1}^m u_{,j}^k u_{,j}^k$$

Proof. Substitute $v = w$ into equation (1) □

Now, we have that it holds that

$$\sup_{k,i,j,l} |b_{ij,l}^k| \leq \sup_{k,i,j,l} |b_{ij,l}^k(0)| + \sup_{k,i,j,l} \left\| b_{ij,l}^k \right\|_{C^{1,\gamma}} (\|u\|_\infty)^\gamma = C_1$$

$$\sup_{k,i,j} |b_{ij}^k| \leq C_2$$

In particular by substitution:

Lemma 4. *It holds that*

$$\sup_l |a_l^k| \leq C_3 \eta^2 \sum_{i=1}^n u_{,k}^i u_{,k}^i = C_3 \eta^2 |u_{,k}|^2 \leq C_3 \eta^2 |\nabla u|^2$$

and

Lemma 5. *It holds that* $\sup_{k,\alpha} |c_\alpha^k| \leq C_3 \eta^2 |\nabla u|$

where $C_3 = C_3(\|u\|_\infty, \gamma, \sup_{k,i,j,l} \left\| b_{ij,l}^k \right\|_{C^{1,\gamma}}, \sup_{k,i,j,l} |b_{ij,l}^k(0)|)$. On the other hand this yields

Proposition 6. *It holds that*

$$\int_\Omega \eta^2 |D_i^h \nabla u|^2 \leq C_4 \int_\Omega \eta^2 |\nabla u|^2 |D_i^h u|^2 dx$$

Proof. It is immediate that

$$\int_\Omega \eta^2 \sum_{j=1}^m |D_i^h u_{,j}^k|^2 \leq \int_\Omega |a_l^k D_i^h u^l D_i^h u^k| + |c_\alpha^k D_i^h u_{,k}^\alpha D_i^h u^k|$$

Now,

$$\int_\Omega |a_l^k D_i^h u^l D_i^h u^k| \leq \int_\Omega C_3 \eta^2 |\nabla u|^2 \sum_{l=1}^n |D_i^h u^l D_i^h u^k| \leq \int_\Omega C_3 c_1 \eta^2 |\nabla u|^2 |D_i^h u|^2$$

and

$$\int_\Omega |c_\alpha^k D_i^h u_{,k}^\alpha D_i^h u^k| \leq \int_\Omega C_3 \eta^2 |\nabla u| |D_i^h u_{,k}^\alpha| |D_i^h u^k| \leq \frac{c_{2,\varepsilon}}{n} \int_\Omega \eta^2 |\nabla u|^2 |D_i^h u^k|^2 + \frac{\varepsilon}{n} \int_\Omega |D_i^h u_{,k}^\alpha|^2$$

and finally

$$|D_i^h u_{,k}^\alpha|^2 \leq |D_i^h \nabla u|^2$$

Then adding them we find that

$$\int_\Omega \eta^2 \sum_{j=1}^m |D_i^h u_{,j}^k|^2 \leq (1 + \frac{c_{2,\varepsilon}}{n}) \int_\Omega C_3 c_1 \eta^2 |\nabla u|^2 |D_i^h u|^2 + \frac{\varepsilon}{n} \int_\Omega |D_i^h \nabla u|^2$$

In particular by taking $\varepsilon = \frac{1}{2}$ and bringing the second term over to the right side we get

$$\int_\Omega \eta^2 |D_i^h \nabla u|^2 \leq 2(1+\frac{c_{2,\frac{1}{2}}}{n})\int_\Omega C_3 c_1 \eta^2 |\nabla u|^2 |D_i^h u|^2$$

□

We must now estimate $\eta^2|\nabla u|^2|D_i^h u|^2$ through the use of the original system.
Let $B = B_{2R}(y)$ for some $y \in \Omega$ with $d(x, \partial\Omega) > \max\{2h, R\}$ and let $v = \eta^2(u - u_B)|D_i^h u|^2$. We adjust η so it has support contained in $B_{2R}(y)$ and is 1 on B_R and pick R possibly even smaller so that $osc_B(u) \leq \frac{1}{2}$.
Then we get

$$v_{,j} = 2\eta\eta_{,j}(u-u_B)|D_i^h u|^2 + \eta^2 u_{,j}|D_i^h u|^2 + 2\eta^2(u-u_B)D_i^h u \cdot D_i^h u_{,j}$$

Thus we get

Lemma 7. *It holds that*

$$\int_\Omega \sum_{i=1}^m u_{,i}^k v_{,i}^k = \int_\Omega \sum_{j=1}^m 2\eta\eta_{,j}(u_{,j}^k(u-u_B)^k)|D_i^h u|^2 + \int_\Omega \sum_{j=1}^m \eta^2 |u_{,j}^k|^2 |D_i^h u|^2 dx$$

$$+ \int_\Omega \sum_{j=1}^m 2\eta^2(u_{,j}^k(u-u_B)^k) D_i^h u \cdot D_i^h u_{,j}$$

This implies immediately that

Proposition 8. *It holds that*

$$\int_\Omega |\nabla u|^2 |D_i^h u|^2 \leq K_1 osc_B(u) \int_\Omega \eta^2 |D_i^h \nabla u|^2 + K_2 \int_\Omega \eta |\nabla\eta|^2 |D_i^h u|^2$$

where $K_1 = K_1(m, n, \sup_{k,i,j}|b_{ij}^k|)$ *and* $K_2 = K_2(R, n, \|u\|_\infty)$

Proof. We have that

$$|\int_\Omega \sum_{i=1}^m u_{,i}^k v_{,i}^k| \leq \int_\Omega |B^k|\eta^2|u-u_B||D_i^h u|^2 \leq mn \sup_{k,i,j}|b_{ij}^k| \int_\Omega |\nabla u|^2 \eta^2 |u-u_B||D_i^h u|^2$$

and we will estimate the three terms in the prior lemma.
First off, since

$$\eta^2|u-u_B| \leq \frac{1}{|B|}\int_B \sup_{y\in B}|u(y)-u(x)| = osc_B(u) \leq \frac{1}{2}$$

it holds that

$$\int_\Omega |2\eta^2(u_{,j}^k(u-u_B)^k)D_i^h u \cdot D_i^h u_{,j}| \leq \frac{1}{2}\int_\Omega \eta^2|\nabla u|^2|D_i^h u|^2 + \frac{1}{2}\int_\Omega \eta^2 |u-u_B|^2 |D_i^h u_{,j}|^2$$

$$\leq \frac{1}{4n}\int_\Omega \eta^2 |\nabla u|^2 |D_i^h u|^2 + C_n osc_B(u)\int_\Omega \eta^2 |D_i^h \nabla u|^2$$

Now, clearly since

$$|\nabla u| \leq \frac{1}{2}(1+|\nabla u|^2)$$

it holds

$$\int_\Omega 2\eta|\eta_{,j}||u^k_{,j}(u-u_B)^k||D^h_i u|^2 \leq \int_\Omega 2\eta|\nabla\eta||\nabla u||u-u_B||D^h_i u|^2$$
$$\leq \frac{\|u\|_\infty}{C_{R,\eta,n}}\int_\Omega \eta^2|\nabla\eta|^2|D^h_i u|^2 + \frac{1}{4n}\int_\Omega \eta^2|\nabla u|^2|D^h_i u|^2$$

Thus

$$\int_\Omega \sum_{j=1}^m \eta^2|u^k_{,j}|^2|D^h_i u|^2 dx \leq \frac{1}{2n}\int_\Omega \eta^2|\nabla u|^2|D^h_i u|^2 + (C_n + mn\sup_{k,i,j}|b^k_{ij}|)osc_B(u)\int_\Omega \eta^2|D^h_i\nabla u|^2 + \frac{\|u\|_\infty}{C_{R,\eta,n}}\int_\Omega \eta^2|\nabla\eta|^2|D^h_i u|^2$$

so in particular

$$\int_\Omega |\nabla u|^2|D^h_i u|^2 \leq 2n(C_n + mn\sup_{k,i,j}|b^k_{ij}|)osc_B(u)\int_\Omega \eta^2|D^h_i\nabla u|^2 + \frac{2n\,\|u\|_\infty}{C_{R,\eta,n}}\int_\Omega \eta^2|\nabla\eta|^2|D^h_i u|^2$$

□

We finally may bound the second order difference quotients:

Lemma 9. *It holds that there are are constants*

$$A_1 = A_1(m,n,\sup_{k,i,j}|b^k_{ij}|, \sup_{k,i,j,l}|b^k_{ij,l}|, \gamma, \sup_{k,i,j,l}\left\|b^k_{ij,l}\right\|_{C^{1,\gamma}}, \sup_{k,i,j,l}|b^k_{ij,l}(0)|, \|u\|_\infty)$$

and

$$A_2 = A_2(m,n,\gamma, \sup_{k,i,j,l}\left\|b^k_{ij,l}\right\|_{C^{1,\gamma}}, \sup_{k,i,j,l}|b^k_{ij,l}(0)|, \|u\|_\infty, R)$$

such that

$$\int_\Omega \eta^2|D^h_i\nabla u|^2 \leq A_1 osc_B(u)\int_\Omega \eta^2|D^h_i\nabla u|^2 + A_2\left\|\nabla u\right\|_{2;\Omega}$$

and so let $U \subseteq \bar{U} \subseteq \Omega$ so that $\bar{U}$ is compact. We note that $\bar{\Omega}$ is also compact and so $osc_B(u)$ depends only on R, not on y. Take $2R < 2h < d(x,\partial\Omega)$ and shrink it so that $A_1 osc_B(u) < \frac{1}{2}$.
Then it is immediate

Proposition 10. *It holds that for $y \in \bar{U}$ that*

$$\int_{B_R(y)} |D^h_i\nabla u|^2 \leq \frac{1}{(1-A_1 osc_B(u))}A_2\left\|\nabla u\right\|_{2;\Omega}$$

so that in particular $u \in W^{2,2}(B_R(y))$ with $\|u\|_{2,2;B_R(y)} \leq (1+\frac{1}{(1-A_1 osc_B(u))}A_2)\left\|\nabla u\right\|_{2;\Omega}$

and so

Theorem 11. *It holds that $u \in W^{2,2}(U)$ with $\|u\|_{2,2;U} \leq (1+\frac{1}{(1-A_1 osc_{B_R}(u))}A_2)\left\|\nabla u\right\|_{2;\Omega}$*