The Isoperimetric Inequality

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Geometric Variational Problems in General

Two Classes:

1) E,F two spaces $C \subset F^E$, *and* $f : C \to \mathbb{R}$. *Find* $\inf_{x \in C} f(x)$

Ex: Harmonic maps: M,N Riemannian manifolds $C = W^{1,2}(M,N)$, $f(x) = \int_N |dx|^2 \mathrm{d}V_N$

2) E a space, $F \subset 2^E, f : F \to \mathbb{R},$ *find* $\inf_{x \in F} f(x)$ *or critical points*

Ex: Plateau's problem, given a closed curve c in $\mathbb{R}^3$

$E = \mathbb{R}^3, F = \textit{surfaces having boundary } c,\ f(x) = Area(x)$

Can sometimes go 2) $\Rightarrow$ 1).

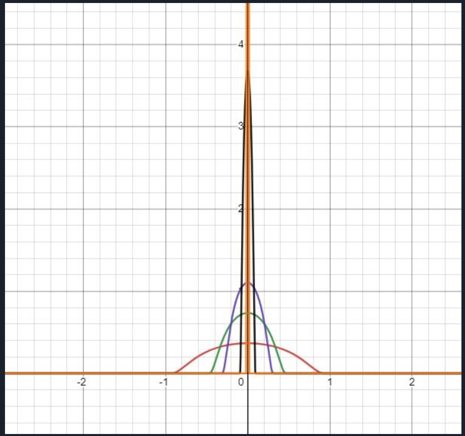
Fundamental Lemma of the Calculus of Variations

Lemma: Suppose $\int_{[0,1]} f\,g\,\mathrm{d}x = 0, \forall g \in C_0^k([0,1]), k \geq 0$

Then f=0

Corollary: This remains true if [0,1] is replaced by a bounded domain in $\mathbb{R}^n$

Corollary: This remains true if [0,1] is replaced by a closed, oriented Riemannian manifold and its volume form



Fundamental Lemma of the Calculus of Variations Pt. 2

Let M be a (closed, oriented)Riemannian manifold with volume form dV. Suppose that $\int_M f\,g\,\mathrm{d}V=0 \quad \forall g \;\; such\ that \int_M g\,\mathrm{d}V=0.$ Then f=const.

Idea:Write $g=h-\frac{\int_M h\,\mathrm{d}V}{\int_M 1\,\mathrm{d}V}$ $for\ h\in C(M)$. $Then \int_M g\,\mathrm{d}V=0$ so with some rearrangement we get $\int_M h\left(f-\frac{1}{Vol(M)}\int_M f\,\mathrm{d}V\right)\mathrm{d}V=0$

Now, the standard fundamental lemma of the calculus of variations implies that

$$f=\frac{1}{Vol(M)}\int_M f\,dV = const$$

The Isoperimetric Inequality in Dimension n

The problem is formulated generally as a class 2) variational problem.

$$E = \mathbb{R}^n, F = bounded\ domains\ with\ C^k\ boundary\ \Gamma(\ of\ ten\ k = 1, 2)$$

$$f = \frac{Area(\Gamma)}{(Vol(F))^{1-\frac{1}{n}}}$$

In this presentation we will want connected boundary and usually a boundary represented as a parametric hypersurface. The problem states that if k>0 then the minimizer is a ball, and this minimizer is unique(for a fixed area or volume).

What is Special in n=2?

In dimension two it holds that the boundary of the unit ball is the circle. Thus since $\mathbb{T}^m = \mathbb{S}^n \Leftrightarrow n = m = 1$ (or =0) it suggests that this is the only nontrivial dimension where we may apply Fourier series.

Similarly, it is the only dimension where the disk is a polydisk. This indicates that complex analysis, and in particular winding numbers and the Cauchy integral formula, may be used.

A Geometric Approach

Let the volume be fixed =V, and then the geometric problem is to minimize $f = \dfrac{Area(\Gamma)}{(Vol(F))^{1-\frac{1}{n}}}$ over domains with connected, twice differentiable, boundary.

We want to show that a minimizer of this functional is a ball with volume V.

If E is a minimizer then $\forall \Phi_t(x) = \Phi(t,x) \in C^{\infty}((-\varepsilon,\varepsilon)\times\mathbb{R}^n,\mathbb{R}^n)$ *such that* $Vol\left(\Phi_t(E)\right) = Vol(E)$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Area\left(\Phi_t(\partial E)\right) = 0$$

The first variation of area of a submanifold with normal vector field v along a vector field X=X(x,t) with corresponding flow Φ(x,t) is $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Area\left(\Phi_t(\partial E)\right) = \int_{\partial E} H\left(X|_{\partial E\times\{0\}}\cdot v\right) \mathrm{d}A$

A Geometric Approach Pt. 2

Theorem: For any $g \in C^2(\partial E)$ *such that* $\int_{\partial E} g \, \mathrm{dA} = 0$ there exists vector fields with flows satisfying the volume preserving property such that $X|_{\partial E \times \{0\}} \cdot v = g$

Corollary: If E minimizes the area-volume functional then it has constant mean curvature

Theorem: If F is a closed, connected hypersurface with constant mean curvature then it is a sphere

Corollary:If E minimizes the area-volume functional then it is a ball

Fourier Series and Derivatives

Let f be a continuously differentiable function on the interval [0,1] with f(0)=f(1), then its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi int}, \hat{f}(n) = \int_0^1 f(s) e^{-2\pi ins} \mathrm{d}s$$

Then it holds that , for n≠0

$$f'(t) = (f')(t) = \sum_{n=-\infty}^{\infty} \widehat{(f')}(n) e^{-2\pi int}$$

$$\widehat{(f')}(n) = \int_0^1 f'(s) e^{-2\pi ins} \mathrm{d}s = (f(s) e^{-2\pi ins}) \Big|_0^1 + 2\pi in \int_0^1 f(s) e^{-2\pi ins} \mathrm{d}s = 2\pi in \int_0^1 f(s) e^{-2\pi ins} \mathrm{d}s$$

More generally, for f as above on the interval [0,L] with f(0)=f(L) it holds that

$$\widehat{(f')}(n) = \frac{2\pi in}{L} \hat{f}(n)$$

Fourier Series and Derivatives Pt. 2

Let f be a continuously differentiable function on the interval [0,1] with f(0)=f(1), then its Parseval's identity implies

$$\|f\|^2_{L^2([0,1])} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

And

$$\|f'\|^2_{L^2([0,1])} = \sum_{n=-\infty}^{\infty} |\widehat{(f')}(n)|^2$$

Wirtinger's Inequality

Assume now in addition that $\widehat{f}(0)=0,\ i.e.\ \int_0^1 f=0$

We have by continuity of f that $\widehat{(f')}(0)=\int_0^1 f'=f(1)-f(0)=0$

It holds then that $\|f'\|_{L^2}^2=\sum_{n\in\mathbb{Z}\setminus\{0\}}|\widehat{f'}(n)|^2=4\pi^2\sum_{n\in\mathbb{Z}\setminus\{0\}}n^2|\widehat{f}(n)|^2$

$$\geq 4\pi^2\sum_{n\in\mathbb{Z}\setminus\{0\}}|\widehat{f}(n)|^2=4\pi^2\|f\|_{L^2}^2$$

Note that if we subtract the left side from the right side of the inequality, we see that equality holds iff all fourier coefficients vanish for |n|>1

Green’s Theorem

Note that by Stokes’ theorem it holds that for $f \in C^{\infty}\left(\mathbb{R}^2, \mathbb{R}^2\right)$

$$\iint_E \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \iint_E Curl(f)\, dxdy = \oint_{\partial E} f_1 dx + f_2 dy$$

Where the last integral is taken in the counterclockwise sense.

Now, if we put f=(-y,x) then $\iint_E \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = 2\iint_E dxdy = 2Area(E)$

$$\oint_{\partial E} f_1 dx + f_2 dy = \oint_{\partial E} -ydx + xdy$$

Isoperimetric Inequality and Fourier Series

We first assume that ∂E can be representing by a simple curve c=(x,y) moving counterclockwise that is: continuously differentiable, c(0)=c(1)(x(0)=x(1) and y(0)=y(1)), of unit length, and parameterized with respect to arclength.

Note that $\int_0^1 x(t)\,dt, \int_0^1 y(t)\,dt$ depends on the position of E, so we may translate it so that both are 0.

Then Green's theorem implies

$$2Area(E) = \oint_{\partial E} -ydx + xdy = \int_0^1 (\dot{y}x - \dot{x}y)\,dt = \int_0^1 c(t)\cdot \vec{n}\,dt$$

$$Thus\ 2Area(E) \le \int_0^1 |c(t)|dt \le \sqrt{\int_0^1 |c(t)|^2 dt} \le \frac{1}{2\pi}\sqrt{\int_0^1 |c'(t)|^2 dt}$$

$$or\ Area(E) \le \frac{1}{4\pi}$$

Note that the last inequality comes from Wirtinger applied to each coordinate function

However, then we must have that, by the equality case in Wirtinger and symmetry, that the overall equality holds iff

$$c(t) = \left(\frac{1}{2\pi}sin(2\pi t + s), \frac{1}{2\pi}cos(2\pi t + s)\right)$$

The Federer-Fleming Theorem

Let $C_{Iso} = inf_{E} \frac{Area(\partial E)}{(Vol(E))^{1-\frac{1}{n}}}$ be the minimum of the area-volume functional, which we know exists now in dimension 2, and define $C = inf_{f \in C_c^{\infty}} \frac{\|\nabla f\|_{L^1}}{\|f\|_{L^{\frac{n}{n-1}}}}$

A theorem of Federer-Fleming states that the two constants are equal.